Seniority Selection Rules of Magnetic Dipole Transitions and Allowed Beta Decay in $1f_{7/2}$ Nuclei

C. NOACK*
Department of Nuclear Physics, The Weizmann Institute of Science, Rehovoth, Israel
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Matrix elements of one-body rank-one tensor operators exhibit seniority selection rules between states of a nuclear j^n configuration. These are given and their influence on β decays and $M1$ transitions in the nuclear $1f_{7/2}$ shell is discussed.

1. INTRODUCTION

IN recent years there has been considerable interest in properties of nuclei in which the $1f_{7/2}$ shell is being filled. One of the reasons for this interest is that binding energies and positions of low-lying excited states in these nuclei are described quite well by a jj-coupling shell model, with the assumption of a pure $(f_{7/2})^n$ configuration.[1] In some cases, the agreement with the experimental data is quite spectacular, so that one is led to believe that the configurations involved are indeed quite pure. Small admixtures to the wave functions, however, will not influence energies, since these are stationary and, therefore, depend quadratically on the admixtures. More stringent tests for the purity of states are provided by considering off-diagonal matrix elements that depend linearly on the admixtures, i.e., some transition probabilities. The purpose of the present paper is to study $M1$ transitions and allowed β decay from this point of view.

The usual classification of the states of a jj-coupling configuration is the seniority classification of Flowers.[2] For maximal isospin, i.e., for identical particles, seniority is a good quantum number for *any* two-body interaction as long as $j\leqslant\frac{7}{2}$.[3] In configurations with both protons and neutrons in unclosed shells this is no longer the case,[4] and, indeed, there is evidence for seniority mixing in the $f_{7/2}$ shell: Eigenfunctions generated by Elliot's procedure[5] (which mixes states of different seniority) have been used by Lawson[6] to calculate properties of these nuclei, and the results are often in much better agreement with experiment than those obtained from the seniority classification.

A direct test for seniority mixing is provided in cases where a transition between states of pure seniorities is forbidden. One example for such a seniority selection rule is the vanishing of $E2$-transition matrix elements between states of maximum isospin and same seniority in the middle of a shell (for $f_{7/2}$ in Ca^{44} and Cr^{52}).[7] In the present paper, we are concerned with the seniority selection rules of transition operators of rank one. These are derived, in general, in Sec. 2. Their application to allowed β decay and $M1$ transitions of some $f_{7/2}$ nuclei are discussed in Secs. 3 and 4, respectively.

2. MATRIX ELEMENTS IN THE j^n CONFIGUATION AND SUM RULES FOR THE COEFFICIENTS OF FRACTIONAL PARENTAGE

A state of the j^n configuration of given symplectic symmetry σ (seniority and reduced isospin[8]), isospin T, and angular momentum J is denoted by

$$\Psi=|j^n\alpha\sigma TZJM\rangle,$$

Z and M are the third components of isospin and spin, and α stands for any additional labels needed to define the state uniquely. The coefficients of fractional parentage[9] (cfp) by which the state Ψ is expressed in terms of the states of the j^{n-1} configuration, are defined by

$$|j^n\alpha\sigma TZJM\rangle=\sum_{\alpha_1\sigma_1T_1J_1}\langle j^{n-1}\alpha_1\sigma_1T_1J_1]\!]j^n\alpha\sigma TJ\rangle \times|j^{n-1}\alpha_1\sigma_1(T_1\tfrac{1}{2})TZ(J_1j)JM\rangle. \quad (1)$$

It has been shown by Racah[10] that they can be broken up into a product of two factors, one independent of J, the other independent of n and T (in the following λ is the label of the representation of $U(2j+1)$, corresponding to n and T, and β,γ are new additional labels):

$$\langle j^{n-1}\alpha_1\sigma_1T_1J_1]\!]j^n\alpha\sigma TJ\rangle =\langle\lambda_1\sigma_1\beta_1]\!]\lambda\sigma\beta\rangle\ (\sigma_1J_1\gamma_1|\sigma J\gamma), \quad (2)$$

where the factors are usually normalized according to

$$\sum_{\lambda_1\sigma_1\beta_1}\langle\lambda_1\sigma_1\beta_1]\!]\lambda\sigma\beta\rangle^2=1,$$

$$\sum_{J_1\gamma_1}(\sigma_1J_1\gamma_1|\sigma J\gamma)^2=1.$$

* Present address: Institut für Theoretische Physik der Universität, Heidelberg, Germany.
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[2] B. H. Flowers, Proc. Roy. Soc. (London) **A212**, 248 (1952).
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[4] It remains true, however, for a more restricted class of interactions; see I. Talmi, Nucl. Phys. **16**, 153 (1960), and K. Helmers, *ibid.* **23**, 594 (1961).
[5] J. P. Elliot, Proc. Roy. Soc. (London) **A245**, 128, 562 (1958).
[6] R. D. Lawson, Phys. Rev. **124**, 1500 (1961); R. D. Lawson and B. Zeidman, *ibid.* **128**, 821 (1962).
[7] See Ref. 3, p. 315.
[8] We shall usually refer to σ as "the seniority," meaning both quantum numbers simultaneously.
[9] For an introduction to the method of cfp, see Ref. 3, p. 243 ff.
[10] G. Racah, Phys. Rev. **76**, 1352 (1949). See also Ref. 2.

TABLE I. List of seniorities σ, σ' for which the diagonality rule (6) exists ($n\leqslant 8$). These are at the same time those for which β decay and $M1$-transition matrices are diagonal (the notation is that of Ref. 2).

n odd	n even			
(1), (1)[a]	(0), (0)[a]	(11), (1111)	(1111), (2211)	(2211), (2222)
(1), (111)	(0), (11)[a]	(11), (22)	(211), (222)	(2222), (2222)
(111), (111)	(11), (11)[a]	(2), (211)	(22), (2211)	
(21), (2111)	(2), (2)[a]	(1111), (1111)	(222), (222)	
(2111), (2221)		(22), (22)		

[a] In these cases the diagonality rule is given already by the J-selection rule of the $6j$ symbol in Eq. (6).

This fundamental property of the cfp is the basis for the considerations below.

We now consider operators of the form

$$\sum_{i=1}^{n} U_{k\mu}(i)\; T_{l\nu}(i)\,,$$

where $U_{k\mu}(i)$ is a tensor of rank k, operating on the space and spin coordinates of the ith particle[11] and $T_{l\nu}(i)$ is a tensor of rank l, operating on the isospin coordinate of the ith particle. The matrix elements of such operators can be expressed as follows[9]:

$$\langle j^n\alpha'\sigma'T'Z'J'M'|\sum_i^n U_{k\mu}(i)T_{l\nu}(i)|j^n\alpha\sigma TZJM\rangle$$
$$=(-)^{T'-Z'}\begin{pmatrix} T' & l & T\\ -Z' & \nu & Z\end{pmatrix}(-)^{J'-M'}\begin{pmatrix} J' & k & J\\ -M' & \mu & M\end{pmatrix}$$
$$\times\langle j^n\alpha'\sigma'T'J'\|\sum_i^n U_k(i)T_l(i)\|j^n\alpha\sigma TJ\rangle\,,$$

with (3)

$$\langle j^n\alpha'\sigma'T'J'\|\sum_i^n U_k(i)T_l(i)\|j^n\alpha\sigma TJ\rangle$$
$$=n\langle j^n\alpha'\sigma'T'J'\|U_k(n)T_l(n)\|j^n\alpha\sigma TJ\rangle$$
$$=\langle\tfrac12\|T_l\|\tfrac12\rangle\,\langle j\|U_k\|j\rangle\times(-)^{J'+T'+j+\frac12+k+l}$$
$$\times[(2J+1)(2J'+1)(2T+1)(2T'+1)]^{1/2}$$
$$\times n\sum_{\alpha_1\sigma_1T_1J_1}(-)^{J_1+T_1}\begin{Bmatrix}\frac12 & T & T_1\\ T' & \frac12 & l\end{Bmatrix}\begin{Bmatrix} j & J & J_1\\ J' & j & k\end{Bmatrix}$$
$$\times\langle j^{n-1}\alpha_1\sigma_1T_1J_1]\!]j^n\alpha\sigma TJ\rangle$$
$$\times\langle j^{n-1}\alpha_1\sigma_1T_1J_1]\!]j^n\alpha'\sigma'T'J'\rangle.$$

We can make use of Eq. (3) to obtain relations among the cfp by inserting for U_k and T_l some special operators whose matrix elements in the j^n configuration are known. Thus, for $U_{k=0}=T_{l=0}=1$, we obtain the orthogonality properties of the cfp:

$$\sum_{\alpha_1\sigma_1T_1J_1}\langle j^{n-1}\alpha_1\sigma_1T_1J_1]\!]j^n\alpha\sigma TJ\rangle\times\langle j^{n-1}\alpha_1\sigma_1T_1J_1]\!]j^n\alpha'\sigma'TJ\rangle=\delta_{(\alpha\sigma)(\alpha'\sigma')}.\quad(4)$$

[11] We tacitly require that $U_{k\mu}(i)$ commute with $\mathbf{j}_i^2$, in order to stay within the framework of jj-coupling.

For $T_{l=0}=1$, $U_{k=1}(i)=\mathbf{j}_i$, we have $\sum_{i=1}^n\mathbf{j}_i=\mathbf{J}$, and, therefore,

$$\langle j^n\alpha'\sigma'TJ'\|\mathbf{J}\|j^n\alpha\sigma TJ\rangle=\delta_{(\alpha\sigma J)(\alpha'\sigma'J')}[J(J+1)(2J+1)]^{1/2},$$

so that we get the sum rule:

$$n\sum_{\alpha_1\sigma_1T_1J_1}(-)^{J_1+j+J+1}\begin{Bmatrix} j & J & J_1\\ J' & j & 1\end{Bmatrix}$$
$$\times\langle j^{n-1}\alpha_1\sigma_1T_1J_1]\!]j^n\alpha\sigma TJ\rangle$$
$$\times\langle j^{n-1}\alpha_1\sigma_1T_1J_1]\!]j^n\alpha'\sigma'TJ'\rangle$$
$$=\delta_{(\alpha\sigma J)(\alpha'\sigma'J')}\left[\frac{J(J+1)}{(2J+1)\cdot j(j+1)(2j+1)}\right]^{1/2}.\quad(5)$$

For $U_{k=0}=1$, $T_{l=1}(i)=\mathbf{t}_i$ we have the analogous sum rule:

$$n\sum_{\alpha_1\sigma_1T_1J_1}(-)^{T_1+T-\frac12}\begin{Bmatrix}\frac12 & T & T_1\\ T' & \frac12 & 1\end{Bmatrix}$$
$$\times\langle j^{n-1}\alpha_1\sigma_1T_1J_1]\!]j^n\alpha\sigma TJ\rangle$$
$$\times\langle j^{n-1}\alpha_1\sigma_1T_1J_1]\!]j^n\alpha'\sigma'T'J\rangle$$
$$=\delta_{(\alpha\sigma T)(\alpha'\sigma'T')}\left[\frac23\frac{T(T+1)}{(2T+1)}\right]^{1/2}.$$

Factorizing the cfp in Eq. (5), we obtain

$$n\sum_{\lambda_1\sigma_1\beta_1}\langle\lambda_1\sigma_1\beta_1]\!]\lambda\sigma\beta\rangle\langle\lambda_1\sigma_1\beta_1]\!]\lambda\sigma'\beta'\rangle$$
$$\times\sum_{J_1\gamma_1}(-)^{J_1+j+J+1}\begin{Bmatrix} j & J & J_1\\ J' & j & 1\end{Bmatrix}$$
$$\times(\sigma_1J_1\gamma_1|\sigma J\gamma)(\sigma_1J_1\gamma_1|\sigma'J'\gamma')$$
$$=\delta_{(\sigma\beta J\gamma)(\sigma'\beta'J'\gamma')}\left[\frac{J(J+1)}{(2J+1)\cdot j(j+1)(2j+1)}\right]^{1/2}.\quad(5')$$

Note that, in general, this holds for $\lambda=\lambda'(T=T')$ only. If, however, for a given σ,σ' there is a set of parameters $\lambda\beta\beta'$ such that there will be *only one* allowed set $\lambda_1\sigma_1\beta_1$ of parent states, then because there is only one term in the $\sum_{\lambda_1\sigma_1\beta_1}$ and the whole expression is diagonal, we must have

$$S\equiv\sum_{J_1\gamma_1}(-)^{J_1+j+J+1}\begin{Bmatrix} j & J & J_1\\ J' & j & 1\end{Bmatrix}$$
$$\times(\sigma_1J_1\gamma_1|\sigma J\gamma)(\sigma_1J_1\gamma_1|\sigma'J'\gamma')$$
$$=\delta_{(\sigma J\gamma)(\sigma'J'\gamma')}f(j,\sigma J\gamma,\sigma_1).\quad(6)$$

[f is a function easily determined from Eq. (5').] This argument can be iterated: Once Eq. (6) is established for one set of seniorities σ,σ', and $\sigma_1=\sigma_{10}$, we can look for another λ such that the $\sum_{\lambda_1\sigma_1\beta_1}$ in Eq. (5') contains

TABLE II. Some β transitions in the $f_{7/2}$ shell influenced by seniority selection rules. Column 1 gives the parent and daughter nucleus, with their respective states, column 2 the possible seniorities of parent and daughter states. Column 3 gives the log *ft* value as calculated from the lowest seniority *italicized* in column 2, and column 4 the experimental log *ft* value. An *F* in column 3 means the transition is seniority-forbidden.

Transition		Possible seniorities	Calc.[a] log *ft*	Experim. log *ft*
$Sc^{42} \to Ca^{42}$	$(0^+ \to 0^+)$	(*0*) → (0)	3.485	3.477[b]
$Sc^{42} \to Ca^{42}$	$(7^+ \to 6^+)$	(*2*) → (11)	4.184	4.185[b]
$Sc^{43} \to Ca^{43}$	$(\frac{7}{2}^- \to \frac{7}{2}^-)$	(*1*), (21) → (1)	3.88	4.9[c]
$Sc^{43} \to Ca^{43}$	$(\frac{7}{2}^- \to \frac{5}{2}^-)$	(*1*), (21) → (111)	*F*	5.0[c]
$Cr^{51} \to V^{51}$	$(\frac{7}{2}^- \to \frac{7}{2}^-)$	(*1*), (21), (2111) → (1)	3.98	5.4[c]
$Cr^{51} \to V^{51}$	$(\frac{7}{2}^- \to \frac{5}{2}^-)$	(*1*), (21), (2111) → (111)	*F*	5.9[c]
$Mn^{52} \to Cr^{52}$	$(6^+ \to 6^+)$	(*11*), (211) → (11)	4.01	5.5[d]
$Mn^{52} \to Cr^{52}$	$(6^+ \to 5^+)$	(*11*), (211) → (1111)	*F*	6.2[d]
$Fe^{53} \to Mn^{53}$	$(\frac{7}{2}^- \to \frac{7}{2}^-)$	(*1*), (21) → (1)	3.88	5.3[c]
$Fe^{53} \to Mn^{53}$	$(\frac{7}{2}^- \to \frac{5}{2}^-)$	(*1*), (21) → (111)	*F*	5.1[c]
$Fe^{53} \to Mn^{53}$	$(\frac{7}{2}^- \to \frac{9}{2}^-)$	(*1*), (21) → (111)	*F*	4.9[c]

[a] The β-decay coupling constant in Eq. (7) was taken from the decay of O^{14} [J. W. Butler and R. O. Bondelid, Phys. Rev. **121**, 1771 (1961)].
[b] P. C. Rogers and G. E. Gordon, Phys. Rev. **129**, 2653 (1963).
[c] *Nuclear Data Sheets*, compiled by K. Way *et al.* (Printing and Publishing Office, National Academy of Sciences–National Research Council, Washington 25, D. C.)
[d] R. R. Wilson, A. A. Bartlett, J. J. Kraushaar, J. D. McCullen, and R. A. Ristinen, Phys. Rev. **125**, 1655 (1962).

only terms with σ_{10} (which vanish) and *one* other term, with σ_{11}, from which we find Eq. (6) to hold for σ_{11}, too. In general, this procedure will show that the diagonality rule (6) exists for all allowed values of σ_1 once it is true for one of them.

A list of the seniorities σ,σ' for which Eq. (6) holds is given in Table I. It contains all those that appear in cfp with $n \leqslant 8$. It is to be noted that the existence of the rule depends on σ,σ' only and not on j.

3. BETA DECAY

The *ft* value for allowed β decay is given by

$$(ft)^{-1} = G^2(\langle 1\rangle^2 + 1,\ 4\langle\sigma\rangle^2)\,, \qquad (7)$$

where $\langle 1\rangle^2$ and $\langle\sigma\rangle^2$ are the Fermi and Gamow-Teller matrix elements, respectively. In our representation (good isospin) the former is simply[12]

$$\langle 1\rangle^2 = \delta_{(TJ\sigma\alpha)(T'J'\sigma'\alpha')}[T(T+1) - Z\cdot Z']\,,$$

while the Gamow-Teller matrix element is given by [compare Eq. (3)]

$$\langle\sigma\rangle^2 = 3(2J'+1)(2T+1)(2T'+1)$$

$$|\langle j\|\sigma\|j\rangle|^2 \begin{pmatrix} T & T' & 1 \\ Z & -Z' & \pm 1 \end{pmatrix}^2 |\Sigma|^2\,,$$

$$\Sigma \equiv n \sum_{\alpha_1\sigma_1T_1J_1} (-)^{J'+J_1+j+1} \times \begin{Bmatrix} J & J' & 1 \\ j & j & J_1 \end{Bmatrix} (-)^{T'+T_1-\frac{1}{2}} \begin{Bmatrix} T & T' & 1 \\ \frac{1}{2} & \frac{1}{2} & T_1 \end{Bmatrix}$$

$$\times \langle j^{n-1}\alpha_1\sigma_1T_1J_1]\!] j^n\alpha\sigma TJ\rangle \times \langle j^{n-1}\alpha_1\sigma_1T_1J_1]\!] j^n\alpha'\sigma'T'J'\rangle\,.$$

[12] Unprimed and primed quantum numbers refer to the initial and final state, respectively.

Factorizing the cfp, we find that

$$\Sigma = n \sum_{\lambda_1\sigma_1\beta_1} (-)^{T'+T_1-\frac{1}{2}} \begin{Bmatrix} T & T' & 1 \\ \frac{1}{2} & \frac{1}{2} & T_1 \end{Bmatrix} \times \langle \lambda_1\sigma_1\beta_1]\!]\lambda\sigma\beta\rangle\langle\lambda_1\sigma_1\beta_1]\!]\lambda'\sigma'\beta'\rangle S\,,$$

where S is the sum in Eq. (6). Thus, the Gamow-Teller transition matrix is diagonal whenever (6) applies, i.e., for the seniorities listed in Table I.

The applicability of these selection rules can best be studied when either the parent or the daughter state has maximum isospin, since in that case its seniority is usually determined by its spin. Examples of such β decays are given in Table II. In the odd-A nuclei, we see that the only way an allowed β decay could proceed is by a $\sigma = (21)$ admixture to the $\sigma = (1)$ wave function of the parent nucleus. However, the admixtures needed to give the experimental branching ratios turn out rather large and, in any case, must be different in Sc^{43} and its hole conjugate Fe^{53}. Thus, rather than choosing another representation (such as that of Ref. 6), the author is inclined to ascribe these discrepancies to admixtures of other configurations.[13] In the case of Mn^{52}, no definite conclusions can be drawn, since the assignment of $(f_{7/2})^4$, $\sigma = (1111)$ to the 5^+ state in Cr^{52} has not been experimentally verified.

The Sc^{42} decay. We have included in Table II the decay $Sc^{42} \to Ca^{42}$. Although the selection rules we are discussing have no direct bearing on this case, it is of particular interest. The states involved in both decays are all uniquely defined by their spins (only one possible seniority), and it is surprising to see that in this

[13] *Note added in proof.* After this paper was submitted for publication, a calculation was published that is relevant to this point [B. F. Bayman, J. D. McCullen, and Larry Zamick, Phys. Rev. Letters **11**, 215 (1963)].

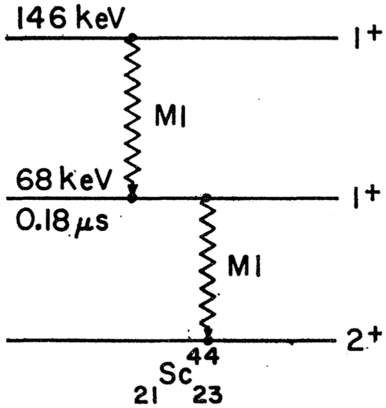

FIG. 1. Partial level scheme of Sc^{44} [from *Nuclear Data Sheets* compiled by K. Way *et al.* (Printing and Publishing Office, National Academy of Sciences –National Research Council, Washington 25, D. C.) and J. K. Kliwer, J. J. Kraushaar, A. A. Bartlett, and R. A. Ristinen, Bull. Am. Phys. Soc. **7**, 341 (1962)].

case the theoretical predictions and the measured values coincide within the experimental errors. It becomes even more surprising when one considers that existing evidence indicates quite strong configuration mixing in the levels of Ca^{42}. This problem is treated in detail elsewhere.[14]

4. *M*1 TRANSITIONS

The effective magnetic-dipole operator in the j^n configuration can be written as

$$(M1)_{\mathrm{op}}=\sum_i^n g_i\mathbf{j}_i=\frac{g_p+g_n}{2}\mathbf{J}+\frac{g_p-g_n}{2}\sum_i^n \tau_3^{(i)}\mathbf{j}_i,$$

where g_p and g_n are the effective g factors of a proton and a neutron in the j^n configuration, respectively. This is completely analogous to β decay, with the role of spin and isospin interchanged. The first term is diagonal ("Fermi"), the second term is a sum of rank 1 tensor operators ("Gamow-Teller"). It is, therefore, immediately clear that this $M1$ operator is again diagonal for all seniorities of Table I. A special case of this rule is the well-known theorem that $M1$ transitions are forbidden in pure configurations of identical particles. Effects of this rule have been observed experimentally in a number of cases, e.g., the decay of the first excited state of V^{51} by an $M1$ transition which is retarded by a factor of about 10^3.[15] But even for configurations of both protons and neutrons there may be cases where the assumption of lowest possible seniority in the lowest lying states implies that $M1$ transitions are forbidden. Unfortunately, there is as yet little information on $M1$ transitions for these nuclei. One possible case of application of our seniority selection rules would be the $M1$ transitions in Sc^{44} (see Fig. 1). There are three $J=1^+$, $T=1$ states in the $(\frac{7}{2})^4$ configuration, one with $\sigma=(2)$, the other two with $\sigma=(211)$. If the 68-keV state were pure $\sigma=(2)$, then the $M1$ transition to it from the 146-keV state would be forbidden. The lifetime of the 146-keV state has not been measured up to now, but we should not expect much retardation on account of our selection rule since the two states lie so close and therefore can be appreciably mixed.

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[14] C. Noack, Phys. Letters **5**, 276 (1963).

[15] I. Y. Krause, Phys. Rev. **129**, 1330 (1963).